# Small Objects in Categories of Algebras
(English abstract)

Jan Jürjens, LFCS, University of Edinburgh

Feb. 1999

## 1 Introduction

This Diplom thesis [J98a] was written in the research group "Categorical Methods in Algebra and Topology (KatMAT)" at the University of Bremen in 1998. In the setting of categorical universal algebra/ categorical model theory it examines algebraic and categorical smallness conditions on objects in categories of algebras and the relationships between them as considered e. g. in [GU71, BH76, AR94].

Parts of this thesis are being published in [J98b].

## 2 General Setting and Motivation

### 2.1 Smallness conditions

Early examples of smallness conditions on objects in categories of algebras (though not using this terminology) are implicit in the concept of a presentation of an abstract group founded in the work of Walther von Dyck (1856-1934):

A presentation of a group $G$ consists of generating elements $\{x_i : i \in I\}$ and generating equations $t_j = s_j$ $(j \in J)$ between terms constructed from the $x_i$ using the group operations. $G$ then consists of the terms in $x_i$ identified by (the congruence hull of) the equations $t_j = s_j$. Every group has such a presentation. Of special importance are those groups with a presentation where the indexing sets $I$ and $J$ are finite ("finitely presented groups"): Every group can be constructed from such groups via "directed colimits". Analogous remarks apply for a slight relaxation of finitely presentedness to "finitely generatedness", where only $I$ has to be finite.

To exploit these two useful notions in other areas of mathematics one can translate it into the categorical language via the use of free objects and coequalizers: An object $A$ in a category is finitely presented iff it is the coequalizer object of a diagram whose two objects are both free objects on finite sets (where a free object on a set $X$ (relative to a forgetful functor $U$ with left adjoint $F$) is $FX$).

It then also makes sense to generalize "finitely-presentedness" to "$\lambda$-presentedness" (and simlarly for finitely generated) for any regular cardinal $\lambda$ so that one can consider infinitary algebras, e. g. domains (this imposes no further difficulty, because the definition of regularity for a cardinal captures exactly what is needed of "finiteness" in the more specific case).

A different notion of smallness of objects, which in varieties (categories of algebras defined by equations) coincides with the previous one, is called "$\lambda$-presentability". It was defined independently in [GU71, AGV72]. An object $A$ in a category $\mathbf{A}$ is "$\lambda$-presentable" if the corresponding covariant hom-functor $\hom(A, \_) : \mathbf{A} \to \mathbf{Set}$ preserves colimits of diagrams whose domains are small $\lambda$-directed categories. This notion has the advantage of being independent of any forgetful functors to **Set** (in contrast to the use of free objects in the definition of "$\lambda$-presentedness"). Also, it is the notion which is fundamental to the theory of "locally presentable categories" (see below).

Similarly, an object $A$ is "$\lambda$-generatable" (in order to distinguish the various smallness conditions we need to use slightly unorthodox terminology here), if the hom-functor $\hom(A, \_) : \mathbf{A} \to \mathbf{Set}$ preserves colimits of diagrams whose domains are small $\lambda$-directed categories whose morphisms are all monomorphisms.

There are various other smallness conditions which prove useful in different contexts and all coincide in the case where the category in question is a variety. The twelve most important of these are examined in this thesis, and for each two of them it is determined, under which conditions an object in a category of algebras fulfills one of them whenever it fulfills the other.

## 2.2 Categories of algebras

In direct generalization of and to unify the situations in the many "algebraic" categories that are commonly considered (like groups, vector spaces, lattices, ...), in universal algebra one considers categories $\mathbf{Alg}\,\mathbf{\Sigma}$ of $\Sigma$-algebras for signatures $\Sigma$ consisting of operations of given arities. A $\Sigma$-algebra is a set together with operations on it as specified in $\Sigma$. Thus one can view e. g. a group as a $\Sigma$-algebra where $\Sigma$ consists of the binary multiplication, the unary inverse element operation and the nullary (constant) neutral element.

For the use in theoretical computer science (in particular structural aspects such as syntax and semantics, data abstraction etc. [Wec92]) it is convenient to consider slightly more general categories $\mathbf{Alg}_{\mathcal{S}}\,\mathbf{\Sigma}$ of $\mathcal{S}$-sorted $\Sigma$-algebras for some sort $\mathcal{S}$. And as above one can in an entirely straightforward manner generalize from algebras with only finitary operations to those with operations whose arities are bounded by a regular cardinal $\lambda$.

Many of the usual categories can be described as subcategories of $\mathbf{Alg}_{\mathcal{S}}\,\mathbf{\Sigma}$

(for suitable $\mathcal{S}$ and $\Sigma$) that are defined by certain equations (like groups are defined via the associativity, the inverse element and the neutral element laws). The famous Birkhoff theorem (the starting point of universal algebra) asserts that a subcategory of some $\mathbf{Alg}_{\mathcal{S}}\,\mathbf{\Sigma}$ is definable by equations iff it is closed under subalgebras, homomorphic images and products in $\mathbf{Alg}_{\mathcal{S}}\,\mathbf{\Sigma}$.

Others (e. g. the category of graphs) are not definable by equations, but more generally by implications of a certain form. These are called "quasivarieties". Again for other categories (e. g. the category of partially ordered sets) one needs to consider "$\lambda$-orthogonal subcategories" of categories of algebras.

While the second kind of categories has a Birkhoff-type closure characterization, an analogous characterization for the latter kind given in [AR94] was disproved in this thesis. However, all three kinds can be characterized categorically (without mentioning the concepts of algebras etc. ). In the latter case, one obtains the result that a category is equivalent to a $\lambda$-orthogonal subcategory of a category of $\lambda$-ary algebras iff it is locally $\lambda$-ary presentable, a natural notion defined in a quite different context in [GU71], which in itself has uses e. g. in domain theory [Plo99] and Linear Logic [Bar91].

To obtain refined results on the interrelations between different smallness conditions on objects in this thesis the above three properties of subcategories of categories of algebras are broken down into a hierarchy of altogether eleven different properties.

## 3 Results

In this thesis it is determined for each two of the twelve smallness conditions and for each of the eleven kinds of categories, whether one of the conditions in that setting always implies the other (where four cases of minor importance remained unsettled), thus determining the minimal conditions (w. r. t. the chosen hierarchy) for such an implication to hold.

The main result is the settling of a problem of almost 30 years standing [GU71]: By giving a counterexample it is shown that a $\lambda$-presentable object in a locally $\lambda$-presentable category with a $\lambda$-presentable regular generator $\mathcal{G}$ need not have a presentation as the coequalizer object of a diagram whose two objects are $\lambda$-small coproducts in $\mathcal{G}$.

From this (and by proving a generalization of the equivalence between $\lambda$-presentedness and $\lambda$-presentability of objects in $\lambda$-orthogonal subcategories of categories of $\lambda$-ary algebras) a theorem in [AR94] is disproved (see above).

The above-mentioned main results are also being published in [J98b]. The thesis is 132 pages long and contains an extensive bibliography of about 250 references.

## References


[AGV72] M. Artin, A. Grothendieck, and J.L. Verdier. *Théorie des topos et cohomologie étale des schémas*, volume 269 of *Springer Lecture Notes in Mathematics*. Springer Verlag, 1972.

[AR94] J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*, volume 189 of *London Math. Soc. Lect. Note Ser.* Cambridge University Press, 1994.

[Bar91] Michael Barr. Accessible categories and models of linear logic. *J. Pure Appl. Algebra*, 69(3):219–232, 1991.

[BH76] B. Banaschewski and H. Herrlich. Subcategories defined by implications. *Houston J. Math.*, 2(2):149–171, 1976.

[GU71] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*, volume 221 of *Lecture Notes in Mathematics*. Springer Verlag, 1971.

[J98a] Jan Jürjens. Kleine Objekte in Kategorien von Algebren (Small objects in categories of algebras). Diplom thesis, KatMAT (Categorical Methods in Algebra and Topology), University of Bremen, 1998. Available (incl. English abstract) at http://www.dcs.ed.ac.uk/~jan/.

[J98b] Jan Jürjens. On a problem of Gabriel and Ulmer. *submitted to Journal of Pure and Applied Algebra*, 1998. available at http://www.dcs.ed.ac.uk/~jan/.

[Plo99] Gordon Plotkin. Lecture on domain theory, 1999. LFCS, Division of Informatics, Univ. of Edinburgh.

[Wec92] W. Wechler. *Universal Algebra for Computer Scientists*. 1992.